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What is questioning the world? Towards an epistemological and curricular break

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To get started: On "the didactic"

A "gesture" δ is looked upon by certain "instances" $\hat{w}$ as *didactic* towards a certain object σ (or set $\mathcal{O}$ of objects σ), with respect to certain persons $x \in X$, if, once this gesture has been performed, the knowledge possessed by $x \in X$ of $\sigma \in \mathcal{O}$ is judged by $\hat{w}$ to be greater than the knowledge possessed by x prior to the performance of δ.

What is “knowing” according to the ATD?

• In the ATD, “the knowledge that the person x has of the object o” is nothing else than the *relation* of x to o, denoted by $R(x, o)$.

• From the point of view of the instance $\hat{w}$, x knows the object o if $\hat{w}$ judges the relation of x to o to be non-empty: $\hat{w} \vdash R(x, o) \neq \emptyset$.

- A person x is, according to $\hat{w}$, an "admissible" subject of an institutional position $\hat{\imath} = (I, p)$ if, for any object σ that $\hat{\imath}$ knows according to $\hat{w}$, i.e., such that

$$\hat{w} \vdash R(\hat{\imath}, \sigma) \neq \emptyset,$$

we have

$$\hat{w} \vdash R(x, \sigma) \cong R(\hat{\imath}, \sigma),$$

which means that $\hat{w}$ judges the relation of x to σ to be "not very different", from the relation $R(\hat{\imath}, \sigma)$ judged to be "ideal" by $\hat{w}$ of anyone who would occupy the position $\hat{\imath}$.

- A person x can be judged to have "good knowledge" of an object σ, i.e.,

$$\hat{w} \vdash R(x, \sigma) \cong R(\hat{\imath}_1, \sigma),$$

which will allow this person to occupy a certain institutional position $\hat{\imath}_1$, but at the same time appear to have insufficient knowledge of σ to occupy a position $\hat{\imath}_2 \neq \hat{\imath}_1$, i.e.,

$$\hat{w}' \vdash R(x, \sigma) \ncong R(\hat{\imath}_2, \sigma).$$

- It follows from this that, in order to fully access the position $\hat{\imath}_2$, x will have to resume studying σ so that we finally have:

$$\hat{w}' \vdash R(x, \sigma) \cong R(\hat{\imath}_2, \sigma).$$

- An example: Student teachers x who had (at least) a bachelor's degree in mathematics (and therefore knew the theory of real number series "well"), however knew nothing about the decimal development of a real number, a notion that, it seems, a mathematics teacher should be familiar with.

- As a student educator, I also considered that these students, who certainly “knew” the concept of a polygon, should also have “some knowledge” of the theorems relating to the *equidecomposability* of polygons. (To the reader who does not “know” what these theorems are, who has never heard of the theorem of Hadwiger-Glur (1951), I simply suggest to *inquire* into this question.)

Principal vs. auxiliary didactic systems

• When a gesture δ relating to an object σ and a set X of persons x, is performed, *learning* will only take place if *didactic systems* $S(X, Y, \sigma)$ appear and function.

• Given any object, three major questions arise: what is the object made of? (its *structure*); how does it work? (its *functioning*); what can it be used for? (its *utility*).

- The notation $S(X, Y, \sigma)$ is a coarse but basic *model* of the *structure* of a didactic system. It is, more precisely, a *modelling tool*.
- An example not yet obsolete: A secondary school class $[X, y]$ is about to study an object σ, which is assumed here to be a *work* that is *not* a question (a question is also, of course, a work, that is to say, a *human production*).

• How will the didactic system $\mathcal{S} = S(X, y, \mathcal{O})$ work? A general answer is as follows: the didactic system $\mathcal{S}$ must work to produce an answer to each of the three major questions relating to the structure of $\mathcal{O}$, its functioning and its utility.

• For each of these three questions Q, the system $\mathcal{S} = S(X, y, \mathcal{O})$ must provide an answer $A = A^{\heartsuit}$ to Q as indicated by the *Herbartian schema*, $S(X, y, Q) \Rightarrow A^{\heartsuit}$.

• How can the answer $A^\heartsuit$ be produced by $\mathcal{S}$? Here is an often-criticized traditional technique. The professor y gives a lecture on ϖ in which he presents, among other things, his/her answer, A_y, to question Q. This answer will be *the* answer $A^\heartsuit$ of the class $[X, y]$, that is to say the answer to Q that students $x \in X$ will henceforth have to "know" and to use in the framework of $[X, y]$.

- This minimalist description of a possible functioning of $S(X, y, \sigma)$ already allows us to underline three essential aspects (which we will find in all cases). First of all, what y does can be modelled as follows: y creates a *didactic milieu* M by introducing a (first) element: (the text of) his/her lecture. From then on, the Herbartian schema will be written as follows: $[S(X, y, Q) \Rightarrow M] \hookrightarrow A^{\heartsuit}$.

- The didactic system $S(X, y, Q)$ is indeed "constructing" a milieu M in order to use it to produce the answer $A^\heartsuit$.
- Here, things are simplified to the maximum. In $S(X, y, Q)$, *only* the teacher y contributes to M, and that moreover through a single gesture—the oral presentation of y's "lecture" on Q (which is a part of y's course on the object σ).

- This may be supplemented by providing students with a written version of his or her oral presentation, or simply by allowing students to take written notes during *y*'s presentation (which in medieval universities, before the end of the 12th century, was not allowed).

- One key notion of the ATD is that of *topos* (from Greek τόπος "place, location"): the word refers to the set of *types of tasks* that a person in a given position may be required to perform (in compliance with this position's praxeological equipment).
- In a didactic system $S(X, y, Q)$, there are thus the *topos* of the "student" $x \in X$ and the *topos* of the "teacher" y.

- In the case mentioned so far, the student *topos* does not allow students to have a say in the elaboration of the didactic milieu *M*, which would be illegitimately encroaching on the teacher *topos*.

- Generally speaking, examining the *topos* of different institutional positions is a key element in the analysis of the functioning of a didactic system—and, indeed, of *any* institution.

• The presentation by y of a lecture on Q presupposes, before the functioning of $S(X, y, Q)$ takes place, the functioning of *another* didactic system that can be written as $S(y, \emptyset, Q)$, whose output will be the answer A_y which will then be presented to X by y:

$$S(y, \emptyset, Q) \Rightarrow M_y] \hookrightarrow A_y.$$

- If we look at $\mathcal{S} = S(X, y, Q)$ as the *principal* didactic system (PDS), the system $S(y, \emptyset, Q)$ is an *auxiliary* didactic system (ADS) of $\mathcal{S}$. There may also exist *induced* didactic systems (IDS): to develop A_y, y may work with other teachers y' and y'' within $S(\{y, y', y''\}, \emptyset, Q)$; or y may do so under the supervision of "trainers" z and z', so that the didactic system $S(Y, Z, Q)$ will then function, where $Y = \{y, y', y''\}$ and $Z = \{z, z'\}$.

- The same is obviously true on the student's side: "homework", which normally gives rise to the formation and functioning of an *auxiliary* didactic system $S(x, \emptyset, Q)$, sometimes also gives rise to *induced* didactic systems, of the form $S(x, \check{y}, Q)$, where $x \in X$ and where $\check{y}$ is a "study helper" (father or mother, older brother or sister, private teacher, etc.), or, more generally, of the form $S(\{x, x', x''\}, \check{Y}, Q)$.

Questioning the world?

- The type of didactic systems examined so far may seem to some people to be “old-fashioned”, archaic, even harmful, in short, whose only future is to be relegated to the museum of outdated pedagogies. From the point of view of the ATD, this would be a gross error.

• Let’s start with a theoretical principle: Any didactic system that can function can have *utility*. And note that what *we* are doing together *hic et nunc* is of the type indicated above in terms of both structure and functioning: a “teacher” y = ϟ (i.e., myself) delivers a presentation in which he attempts to make known an answer $A_{ϟ}$ to a question Q which can be formulated as follows: “What is (the paradigm of) questioning the world?”

- There are similarities between an ordinary secondary school classroom and our conference. In a class, for example, some students are extremely interested in what the teacher is saying, while others, no doubt a minority, are indifferent, to the point where some of them may even wonder what they are doing there... I see no reason why it should be very different for us—hoping, of course, that the uninterested are exceptions.

• However there’s an essential difference. The participants x to this congress who happen to inquire into Q belong to a didactic system $S(\check{X}, \check{Y}, Q)$, where $\check{X} \ni x$ and where $\check{y} \in \check{Y}$ helps in some way with the inquiry into Q (of course we can have $\check{Y} = \emptyset$). In this case, the didactic systems $S(\check{X}, \check{Y}, Q)$ are *principal* didactic systems, while the didactic system we form together here, that is, $S(X, ϟ, Q)$, becomes an *auxiliary* didactic system.

- In truth, when a team $\check{X}$ of students or researchers is engaged in an inquiry into a question Q, that is, in a *study and research process* (SRP_r) relating to Q, which determines a study and research *path* (SRP_a) towards an answer A to Q, didactic systems other than $S(\check{X}, \check{Y}, Q)$, are auxiliary (or induced) didactic systems.

- I can now begin to clarify what the *paradigm of questioning the world* is. Let us start with the bare minimum, namely a didactic system of the form $S(X, \emptyset, Q)$, where Q is a question and X is a non-empty set of "inquirers" (students, researchers, etc.). Immediately, two questions arise.

- The first is: where does the Q question come from? This is an important aspect of what I have called the *destiny of questions* within an institution—a classroom, a laboratory, etc.—or a complex of institutions.

- The second question is: Who are the students $x \in X$, and how did the set X originate?

- These two questions remind us that a didactic system does not live in a social vacuum but in an institutional universe that generates conditions and constraints. In the *scale of levels of didactic co-determinacy* specific to the ATD, we find the following sequence of levels:

…

⇃↾

Society

⇃↾

School

⇃↾

Pedagogy

⇃↾

Didactic system

⇃↾

…

- It may be noted that if we read this diagram as a model of the world of secondary education, one level seems to be missing: that of the *classes*, as referred to above. This absence is voluntary: dividing the population of a school into permanent “classes” is a *pedagogical* provision, which may or may not be implemented (there are, in the case of France and other societies, historical counter-examples).

- It should also be remembered that the word *school* (from the Greek σχολή, *skholè*), as used here, refers to an institution of *any sort* that provides a *place* and *time* that enable its subjects to engage in *study* (or in the direction of study), escaping for a moment from the other obligations of personal and social life.

- The word “school” therefore refers to a very broad spectrum of institutions. In addition to the fact that every family with young children *is* a school (which hosts didactic systems auxiliary to the principal didactic systems formed in the classrooms to which these children belong), it must also be seen that every laboratory or research centre is a school in the sense of the ATD.

- It is the institutional environments of $S(X, \emptyset, Q)$ that will largely determine both X and Q. For example, if $S(X, y, Q)$ is formed in an 8th grade class whose mathematics teacher is y, we know roughly how X was defined.
- In this case, we also know that the question Q can be, for example: "What is the length of the diagonal of a rectangular box whose side lengths are a, b, and c?"

- If the class is a *thematic workshop* offered to 10th graders, it could be—as a possible pedagogical arrangement—that the students $x \in X$ have chosen the workshop on “global warming” (among several other possibilities). In this case, it may be that the question Q is simply stated as follows: “Why do carbon dioxide emissions into the atmosphere contribute to global warming?”

- A third example, related to the first one above, concerns inquiries that teachers may have to conduct on their own. In a lower secondary school, four mathematics teachers, y_1, y_2, y_3, y_4, have set up a fortnightly seminar to explore questions arising from their teaching. This seminar is an institution similar to a classroom, which allows them to escape momentarily from the usual obligations of their lower secondary school.

- For reasons related to their didactic needs as teachers, they then form the didactic system $S(Y, \emptyset, Q)$ where $Y = \{y_1, y_2, y_3, y_4\}$ and where Q is formulated as follows: "For which triplets of strictly positive integers a, b, c is there a strictly positive integer d such that $a^2 + b^2 + c^2 = d^2$?"

- The *question of questions and their destiny* is at the heart of the paradigm of questioning the world. The cardinal principle here is that, for a collective *X* of persons to inquire into a question *Q*, it is essential that this question “arises” for this collective, that is to say, that providing a “good” answer *A* to *Q* satisfies a praxeological need (in terms of *praxis* and *logos*) experienced by *X*.

- A key problem here is: Who decides whether *X* needs an answer to *Q*? Who ultimately decides that *X* should have the means to study *Q*? The three examples above concern three different cases (which, by themselves, do not exhaust the possible cases, of course).

- In the first case, we find a very classic situation, which comes under the *paradigm of visiting works*: there is, in France, a national study programme for the 8th grade which imposes to study “the Pythagorean theorem”. The study of this “prestigious” work *w* involves, often very lightly, a study of its structure, function and utility—it is the latter component that is concerned by the question put to the students *X*.

- The students here have *no part* in the choice of *w* or in the choice of the question of the diagonal of a rectangular box. Tutelary powers (minister, teacher, etc.) decide all this for them.

- In the second case—the four teachers and their fortnightly seminar—the idea of praxeological *need* is much more identifiable.

- One might think that, with regard to the “applications” of the Pythagorean theorem, these teachers would like to offer their students triplets of strictly positive integers (a, b, c) such that $a^2 + b^2 + c^2$ is the *square* of an integer—before moving on to other cases. The need to answer the question Q under study is rooted in this requirement.

- It should be noted that such an inquiry could just as well concern *all* teachers of mathematics in an 8th grade class. But the study of this question, if I am well informed, is not generally regarded as a must (among others) of the profession, a fact one can see as a symptom of the *praxeological abandon* from which the profession suffers.

- The third case is, in a sense, intermediate. Here, we are concerned with getting 10th graders to inquire into a question that is known to everyone but to which it is not so easy to provide an answer, even though it would be likely to satisfy what is widely recognised as a prevalent praxeological need of all citizens.

- Two remarks are in order here. First remark: as a general rule, when one tries to give an answer A to a “complex” question Q, one can only arrive at a *partial* answer. The inquiry makes a more or less developed contribution to the elaboration of a “total” answer, *which may not exist*.

- Second remark: when a didactic system $S(X, Y, Q)$ comes into existence (with possibly $Y = \emptyset$), one must consider as a general law that Q is there because it has *imposed itself on X*, for whatever reason, whether one judges this imposition justified or illegitimate. Students did not choose the question *just because* they would “like” to study it. This, in truth, has important consequences for the destiny of questions.

- In fact, some questions are never asked—at best, they are stillborn—, others will be forgotten very quickly or will fall into epistemological disregard—no one will be found to "care" about them. In the historical process to install in our societies and its schools the paradigm of questioning the world, an essential operation aims to fashion a programme of questions that meets a decisive criterion.

- Here is this criterion: given a school σ (in the broad sense already explained), or a system of schools Σ, the “curriculum of questions” $\boldsymbol{Q}$ of σ (or Σ) must gather the questions which the “student” in a given position *p* must have studied in order to access, equipped with adequate praxeological equipment, a subsequent position *p′* within the school or outside it.

What we are faced with

- This process of creating a *new didactic world* clashes hard with the old world, in which what is prominent are the *works* w to be studied, and in which questions are often a pretext for studying a work w determined a priori. How does this happen?

- In order to study a question *Q*, one needs *tools*, which find in it a part of their utility. In the historical evolution that led to the hegemony of the paradigm of visiting works, tools were put forward for their own sake, and their uses (to answer questions) were, in effect, downplayed. Programmes of study were formulated in terms of *works*. Instead of a programme of questions, programmes of works were imposed.

- Then, in connection with the development of didactics, questions were rehabilitated and sought after as a means of provoking the encounter with works that they then made appear useful if not indispensable to the elaboration of answers to questions. At the same time, works, considered in and of themselves, appeared to be privileged, the questions that motivated their study being generally accessory, as so many pretexts.

- How does this work? The teacher y starts from a work w having a certain curricular “prestige”, and looks for a question Q whose study by X, under y’s supervision, must lead X to meet and study, to a certain extent, the work w. Here, the didactic system $S(X, y, Q)$ is an *auxiliary* didactic system *generating* the *principal* didactic system $S(X, y, w)$.

- By contrast, in the new paradigm, the class $[X, y]$ starts from a question Q, whose study by the principal didactic system $S(X, y, Q)$ generates auxiliary didactic systems of the form $S(X, y, w)$, where the utility of w in the study of Q is key. The study of w and the degree to which it is deepened depend in such a case on the need for knowledge arising from the study of Q, and *not on any teaching tradition*, however venerable.

- Highlighting works while forgetting the questions they help or have helped to answer, is obviously creating an “upside down” world. However, this state of affairs has not been uniformly established from the outset. Here is an example.
- It is usual to teach students this characteristic property of parallelograms: their diagonals bisect each other.

- It seems that today’s mathematics teachers would not know how to answer the question, “What is the point of knowing this?” More than that, I believe that most of them have never asked themselves this question. Faced with it, some would no doubt reply, not without arrogance: “It’s not meant to be used, it just is!” One could retort maliciously: “Yes, like a hammer or a saw…”

- However there were textbooks, still in the 20th century, which explained to their young readers what was the “utility” of parallelograms and in particular of the property of their diagonals. But again I will let you inquire into this question for yourselves!

• Here is a second example. Until the last few years, the curriculum for the 8th grade in France stated that students should know the equality $\frac{a}{b} = a \times \frac{1}{b}$. Like the "property of the diagonals", it is certainly true! But this is not enough: the equality $a \times b = \frac{1}{\frac{1}{a} \times \frac{1}{b}}$, ignored by the curriculum, is just as true for example.

- In fact, the equality $\frac{a}{b} = a \times \frac{1}{b}$ is a curricular vestige of a bygone era, when, lacking "modern" calculation tools, divisions were sometimes avoided and replaced by multiplications. For example, we had $\frac{31}{2} = 31 \times \frac{1}{2} = 31 \times 0.5 = 15.5$, or $\frac{17}{5} = 17 \times \frac{1}{5} = 17 \times 0.2 = 3.4$.

- One had to bear in mind that $\frac{1}{2} = 0.5$, $\frac{1}{5} = 0.2$ or $\frac{1}{3} \approx 0.33$. The most remarkable use of this technique concerned the number π. In order to draw a circle on the ground with a circumference of 50 metres, one has to take a radius $r \approx \frac{50}{2\pi}$ m.

We then have: $\frac{50}{2\pi} = \frac{25}{\pi} = 25 \times \frac{1}{\pi}$.

- At this point, the student must have in mind that $\frac{1}{\pi} \approx 0.32$ (or, more precisely, 0.318), in the same way that we know that $\pi \approx 3.14$ or 3.1415. Here, we would then have: $\frac{25}{\pi} \approx 25 \times 0.32 = 100 \times 0.08 = 8$. (Today, using the Google calculator $\mathbb{G}$, we get $\frac{25}{\pi} =_{\mathbb{G}} 7.95774715459$.)

- In the world of works generated by human activity, by scientific and technical activity in particular, there is a permanent updating of the collections of works on which a given institution feeds its activity. One of the simplest examples relates to quadratic equations. Everyone still learns today, for example, that the equation $ax^2 + bx + c = 0$ (with $ac \neq 0$) has the solutions $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

- However, in the Wikipedia article “Quadratic equation”, one also finds the following non-traditional formula:

$$x = \frac{2c}{-b \pm \sqrt{b^2 - 4ac}}.$$

But what is the purpose of this formula? Here again, I will leave readers to inquire for themselves.

• If, in a general way, the evolution of the questions being addressed in a given field gives rise to *new* tools, this evolution can give new life to *old*, largely forgotten tools. A striking case is that of *computational geometry*, prompted in particular by the needs of *computer graphics*, where the names of mathematicians Georgi Voronoi (1868–1908) or Boris Delaunay (1890–1980) have reappeared.

- Many other examples of unprecedented progress and revisits to old creations could be given. What is important is to emphasise that the key criterion is not a formal one, but a *functional* one: a work *w* interests us because it allows us to construct an answer to a question *Q*, without which life would be a little less appropriate.

Uses and misuses of the didactic transposition

- The historical process of disconnecting works from questions has had and continues to have severe consequences. Indeed, the ever-active processes of *didactic transposition* will take hold of the works, and free them from the *functions* they originally fulfilled. To continue a comparison sketched above, one could think that, in order to “simplify” the teaching of the saw and the hammer, one could replace the blade of the saw with a stretched string or the head of the hammer with a piece of foam.

- This process of simplification of works went hand in hand with a correlative process of "purification" of the mathematics taught, which tended to break their organic links with the phenomena studied in disciplinary fields that were originally close, but from which they seemed to want to keep their distance, as nothing "non-mathematical" should enter the mathematics classroom. To put it bluntly, the mathematics classroom appeared to be cut off from the rest of the world!

• I will quickly illustrate this phenomenon with an example that was the subject of a work co-authored by Heidi Strømskag and myself entitled *Elementary algebra as a modelling tool: A plea for a new curriculum*. What we study in this paper is an essential phenomenon, which has led to the *elimination of parameters from elementary algebra*. What is it about? Consider first the equation $1.2x = 3$. This is one of the simplest equations one can imagine. Its solution is $x = \frac{3}{1.2} = \frac{1}{0.4} = 2.5$.

• The point to note here is that the solution is a *number*. Now consider Ohm's law, which I think everyone knows: $V = RI$. This equality can be viewed as a *first degree equation* with respect to the unknown $x = R$. The solution of the equation is $x = R = \frac{V}{I}$. This time the solution is no longer a number, but a *formula*, namely the formula $R = \frac{V}{I}$. The elimination of parameters in principle prohibits the small algebraic calculation I just did.

• The teacher or textbook will tell the students that, e.g., $V = 3$ volts and $I = 1.2$ amperes, and students will then have to solve the *parameter-free* equation $3 = 1.2R$, which leads to $R = \frac{3}{1.2} = \frac{1}{0.4} = 2.5$. (If you don't eliminate units, you have $R = \frac{3\,\mathrm{V}}{1.2\,\mathrm{A}} = \frac{3}{1.2}\Omega = 2.5\Omega$.) The *vanishing of parameters* leads to the "deparametrization" of formulas, before embarking on "pure" numerical calculations.

• The previous example and many others show that it is difficult to model a system *without* working with parameters. I mentioned above the question “Why do carbon dioxide emissions into the atmosphere contribute to global warming?” One of the earliest papers on this subject dates from 1896: it is entitled *On the Influence of Carbonic Acid in the Air upon the Temperature of the Ground* and is due to Svante Arrhenius (1859-1927), who was awarded the Nobel Prize for Chemistry in 1903 (*Philosophical Magazine and Journal of Science*, Series 5, Volume 41, April 1896, pp. 237–276).

- Here is a very brief extract from this study:

Then we find for the column of air

$$\beta\gamma\theta^4 = \beta\gamma\nu(\mathrm{T}^4 - \theta^4) + \alpha\mathrm{A} + \mathrm{M} \quad (1)$$

The first member of this equation represents the heat radiated from the air (emission-coefficient β, temperature θ) to space (temperature 0). The second one gives the heat radiated from the soil (1 cm.^{2}, temperature T, albedo $1 - \nu$) to the air; the third and fourth give the amount of the sun's radiation absorbed by the air, and the quantity of heat obtained by conduction (air-currents) from other parts of the air or from the ground. (pp. 255–256).

- In 2009, 113 years later, MIT environmental engineering experts Michael Szulczewski and Ruben Juanes published (in *Energy Procedia*, Volume 1, Issue 1, pp. 3307–3314) a study on carbon capture and storage entitled *A simple but rigorous model for calculating CO_2 storage capacity in deep saline aquifer at the basin scale*.

- Here is a very short extract from their study:

There are two important equations of the model. The equation to calculate the storage capacity is:

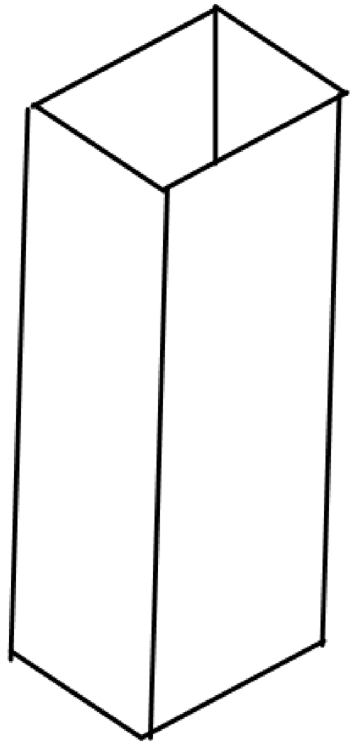
$$C = \frac{2M\Gamma^2(1 - S_{WC})}{\Gamma^2 + (2 - \Gamma)(1 - M + M\Gamma)} \rho_{CO2}\phi HWL_{total}, \quad (1)$$

where C is the mass of trapped CO_2, Γ is the trapping coefficient, M is the mobility ratio, ρ_{CO2} is the density of CO_2, ϕ is the porosity, W is the length of the injection array, H is the net sandstone thickness of the reservoir, and L_{total} is the total extent of the CO_2 plume… (p. 3309)

• This is the main formula of the "simple but rigorous" model that the authors propose. These examples remind us that, when we model a system and do not eliminate parameters by assigning them arbitrary numerical values, parameters swarm... But I would now like to illustrate with a new example what we might want to do with secondary school pupils—within the framework of a "new curriculum" which is still far from existing!

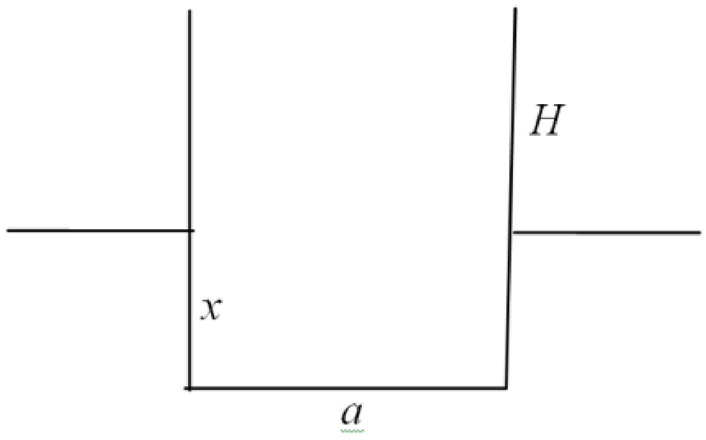
• This example derives from a research activity carried out a long time ago with colleagues from the IREM of Aix-Marseille in a workshop set up in a lower secondary school in Marseille and attended by 9th graders. I reported on the activity of this workshop in a publication of the IREM of Aix-Marseille released in 1989 and entitled *Arithmetic, Algebra, Modelling: Stages of a research study*. Here I will only refer to the physical system considered with the students and its modelling.

- The activity proposed to the students was called “Floating boxes”. A box with a square base, without a lid, is considered and the question of its buoyancy is raised.



- The boxes used were made of lead, so as to create a sense of paradox: a box made of a "heavy" material which may or may not float—for example, there were large boxes that floated and small ones that sank, etc.

- Let's model this type of systems. If the base of the box is a square of side a and the box is of height H, the area of the walls of the box is equal to $a^2 + 4aH$.



• Let us denote by σ the mass per unit area of the walls. The mass of the box is therefore $(a^2 + 4aH)\sigma$. If the box floats by sinking a length x, the buoyancy is equal to $a^2 x\mu$, where μ is the density of the liquid in which the box is immersed. The solution in x of this equation is

$$x = \left(1 + \frac{4H}{a}\right)\frac{\sigma}{\mu}.$$

If $x < H$, the box floats; if $x > H$, the box sinks. Therefore, we need to study this inequality:

$$\left(1 + \frac{4H}{a}\right)\frac{\sigma}{\mu} < H.$$

• If $\gamma = \sigma\mu$; we have $\left(1 + \frac{4H}{a}\right)\gamma < H \Leftrightarrow \left(\frac{1}{\gamma} - \frac{4}{a}\right)H > 1$.

Before going further, two remarks will be useful. Firstly, the liquid used was water, so I will take here $\mu = 1\,\mathrm{g/cm^3}$. I assume that the walls were $3\,\mathrm{mm} = 0.3\,\mathrm{cm}$ thick, so that $1\,\mathrm{cm^2}$ of wall has a mass equal to that of a volume of $0.3\,\mathrm{cm} \times 1\,\mathrm{cm^2} = 0.3\,\mathrm{cm^3}$ of lead. The density of lead is $11.35\,\mathrm{g/cm^3}$; we have: $(11.35\,\mathrm{g/cm^3}) \times 0.3\,\mathrm{cm^3} = (11.35 \times 0.3)\,\mathrm{g} = 3.405\,\mathrm{g}$. I will take $\sigma = 3.4\,\mathrm{g/cm^2}$, which gives $\gamma = \frac{\sigma}{\mu} = \frac{3.4\,\mathrm{g/cm^2}}{1\,\mathrm{g/cm^3}} = 3.4\,\mathrm{cm}$.

• Let us return to the inequality $\left(\frac{1}{\gamma} - \frac{4}{a}\right)H > 1$, which should give us the minimum value of the height H for the box to float. A person working in a hurry might then write:

$$\left(\frac{1}{\gamma} - \frac{4}{a}\right)H > 1 \Leftrightarrow H > \frac{1}{\frac{1}{\gamma} - \frac{4}{a}}.$$

The minimum height would therefore be $H_m = \frac{1}{\frac{1}{\gamma} - \frac{4}{a}} = \frac{a\gamma}{a - 4\gamma}$. But here an “obstacle” appears.

• We became aware of a phenomenon that we had not anticipated: we thought that if a box sank, and we increased its height sufficiently, it would end up floating. This was not experimentally feasible in the classroom—what would we do if H_m was of the order of several metres for example? But what the above algebraic modelling makes clear is that we only have the equivalence $\left(\frac{1}{\gamma}-\frac{4}{a}\right)H > 1 \Leftrightarrow H > \frac{1}{\frac{1}{\gamma}-\frac{4}{a}} = \frac{a\gamma}{a-4\gamma}$ *if* $\frac{1}{\gamma}-\frac{4}{a} > 0$, that is, if we have: $a > 4\gamma$.

• Here, $4\gamma = 4 \times 3.4\,\text{cm} = 13.6\,\text{cm}$. If $a < 4\gamma = 13.6\,\text{cm}$, *whatever its height, the box will sink*. This is a “result” that the study done in the workshop was revealing to us! To put it more loosely, if the box has too small a base, it will *never* float.

A fascination for works always rekindled?

• The example of the floating boxes, like any study of a question *Q*, can revive in each of us the atavistic attraction for *works*, or, more exactly, for works already made, recognized, long taught, and often “prestigious”. (I recall in passing that, originally, “prestigious” means “which makes illusion, is deceitful”.)

• Arrived at the formula $H_m = \dfrac{1}{\dfrac{1}{\gamma} - \dfrac{4}{a}} = \dfrac{a\gamma}{a - 4\gamma}$, some readers will say to themselves: "There, we have a function *f* of the variable *a*, we can study its sign, and for that calculate its derivative, etc." So here we are back in the old world! Denoting by $\overset{s}{=}$ the relation "have the same sign as", and for $a > 4\gamma$, we will thus have $H_m = \dfrac{a\gamma}{a - 4\gamma} \overset{s}{=} a - 4\gamma$, from which we deduce that H_m is strictly positive.

• What happens now when a increases (with, still, $a > 4\gamma$)? We can reason as follows: "When a increases, then $\frac{4}{a}$ decreases, so that $\frac{1}{\gamma} - \frac{4}{a}$ increases and its inverse H_m decreases. Thus, when $a > 4\gamma$ increases, then H_m decreases. Or: when the *base* of the box *increases*, the minimum height for the box to float *decreases*." Here we have just entered the *dialectic of media and milieus*. We can (and must) use multiple milieus.

• Let’s start with a simple *numerical* calculation, done here with Excel:

a	H_m
50	4,7
45	4,9
40	5,2
35	5,6
30	6,2
25	7,5
20	10,6
15	36,4
10	-9,4

a	H_m
15	36,4
14	119,0
13,9	157,5
13,8	234,6
13,7	465,8
13,69	517,2
13,68	581,4
13,67	664,0
13,66	774,1
13,65	928,2
13,64	1159,4
13,63	1544,7
13,62	2315,4
13,61	4627,4

- We can see here that, as the value a decreases, H_m increases. This fact is even more striking when a approaches the limit value 13.6. We can see that experimentation here would require the construction of a box some 5 metres high!

• Let us now turn to *algebraic* calculation. The following calculation provides an answer that is in agreement with the above: if $4\gamma < a < b$, then

$$H_m(a) - H_m(b) = \frac{a\gamma}{a - 4\gamma} - \frac{b\gamma}{b - 4\gamma}$$

$$= \frac{a\gamma(b - 4\gamma) - b\gamma(a - 4\gamma)}{(a - 4\gamma)(b - 4\gamma)}$$

$$= \frac{4\gamma^2}{(a - 4\gamma)(b - 4\gamma)}(b - a) > 0.$$

• Therefore, when *a* increases, H_m decreases: we find again the previous result. Of course, if one has studied *elementary differential calculus*, one can study the variation of the function $f(x) = \dfrac{1}{\dfrac{1}{\gamma} - \dfrac{4}{x}}$ by studying the sign of its derivative. Here we have: $f(x)' \overset{s}{=} -\left(\dfrac{1}{\gamma} - \dfrac{4}{x}\right)'$ $\overset{s}{=} \left(\dfrac{4}{x}\right)' \overset{s}{=} -x' = -1$. The function f is therefore decreasing, as expected.

• I must say, however, that I would rather see beginners in *algebraic* calculus check the validity of the equality established above:

$$\frac{a\gamma(b-4\gamma)-b\gamma(a-4\gamma)}{(a-4\gamma)(b-4\gamma)}=\frac{4\gamma^2}{(a-4\gamma)(b-4\gamma)}(b-a).$$

To do so, we can start by observing that, if $b = a$, the numerator $a\gamma(b - 4\gamma) - b\gamma(a - 4\gamma)$ is zero; it is therefore divisible by $b - a$. What is the cofactor of $b - a$? Let $b = a + 1$; then the numerator is equal to the cofactor. We have $a\gamma(1 + a - 4\gamma) - a\gamma(a - 4\gamma) - \gamma(a - 4\gamma) = a\gamma - \gamma a + 4\gamma^2 = 4\gamma^2$, as expected.

Under the leadership of questions

• In secondary schools, where parameters have, for the most part, disappeared, the little algebraic work explained above could no longer be done today—as it was already the case in the previous decades. In this regard, I refer again to the study by Heidi Strømskag and myself, which tries to show the path to a thorough renovation of the algebraic curriculum.

- The phenomenon is in fact general: as soon as works are no longer in the service of the study of questions, they tend to evolve under the influence of other forces that have their own logic, that of the institutions that provide these works with their habitat.

- When these institutions are didactic institutions, these forces are those of *didactic transposition*, which, by seeking to simplify the “knowledge to be taught” in order to make it more easily teachable, will often distort these works to the point of making them lose much of their genuineness and utility.

- Conversely, all that we know leads us to believe that the *rejuvenation* of curriculums can only take place under the condition of *giving leadership to the questions* to be studied, the works created or studied and used being then organically linked to the study of those questions.

- In conclusion, I will attempt to briefly describe the *core* of the paradigm of questioning the world—a paradigm whose conditions of possibility must be studied by didacticians at *all levels* of the didactic co-determinancy scale. We have seen that a *school* is any institution that offers persons who come to occupy a student position a *time* and a *place* for the study of questions or other works. I will refer generically to a school $\mathcal{K}$.

- A school $\mathcal{K}$ in principle offers a sequence of student positions p_1, p_2, …, p_n, where $n \geq 1$. The curriculum of $\mathcal{K}$ then must specify, for each position p_i ($1 \leq i < $ n), the set $\boldsymbol{\mathcal{Q}}_i$ of questions Q_i a student x must study (satisfactorily) in that position; and, for $i = n$, similarly, the set $\boldsymbol{\mathcal{Q}}_{\text{ex}}$ of questions Q_{ex} the student must study (satisfactorily) in order to gain access, according to $\mathcal{K}$, to positions p_{ex} of other institutions in society.

• One of the great problems in the life of a school $\mathcal{K}$ is the "making" of the sets $\boldsymbol{Q}$ of questions Q and their accreditation by various "guardian" institutions. That accomplished, what students come to do in $\mathcal{K}$ is to study questions Q, that is, to *inquire* into those questions, and, through that, to dispose, then, of supposedly "appropriate" answers A.

- An essential point of the type of didactic scenarios I describe here is that the choice of a question Q_i should depend *solely* on its relevance to the position p_i, and in *no case* on the works that could be relevant tools in the study of this question.

- Some teachers might say for example: “Studying this question inevitably leads to studying the logarithm function, and for that reason it is an interesting question...”. This is typically an “old world” viewpoint, where works, taken *per se*, are everything, and questions are mere pretexts for encountering (and studying) works.

- In the paradigm of questioning the world, by contrast, one must *accept to ignore* what the inquiry into *Q* will reveal (at least partially), namely that some works *w* could be used (that is, could be *tools*) in the inquiry into *Q* in order to lead to an appropriate answer *A*.

- More specifically, five different cases of works may be distinguished:

⓪ the work *w* is known to the students in position *p* and their knowledge of this work is adequate for the current inquiry;

① the work *w* is known to the students in position *p*, but their relation to this work must be partially reworked so that it becomes adequate for the current study;

② the work w is unknown to the students in position p, but its study in order to develop a relation to this work that is adequate for the current study is considered possible under the conditions and constraints of this inquiry;

③ the work w is unknown to the students in position p, its study with a view to developing a relation to this work adequate to the current inquiry is deemed impossible under the conditions and constraints of this inquiry, but the students can understand in an authentic way the use made of w by authors who have themselves studied Q using w and made known their answers $A^\lozenge$;

④ the work *w* is unknown to the students in position *p*, and, under the conditions and constraints of the inquiry, its study with a view to developing a relation adequate to the inquiry is deemed impossible, and the students cannot authentically understand the use that can be made of it in the inquiry in progress—they thus reach the current limits of their questioning of the world on this point!

- In the paradigm of visiting works, a lecture on a work *w* aims to make known successively its *structure*, its *functioning*, and finally (a part of) its *utility*. By contrast, in an inquiry into *Q*, the encounter with *w* *begins* with an encounter with its (partial) *utility* as shown by the intention to use it in this inquiry, as carried out by the students concerned or by authors whose analyses they encountered during their inquiry.

- The study of the work’s functioning, which allows it to be used, comes afterwards, and this is all the more true of the study of its structure. For example, you can use the logarithm function to solve the equation $3^x = 17$ (you get $x = \ln 17/\ln 3 =_{\mathbb{G}} 2.57890192316$) without “knowing everything” about the logarithm function. *This “epistemic minimalism” is one of the golden rules* in *questioning the world*.

- Another remark must be made about the conduct of inquiry. One of the new principles of this conduct can be stated informally as follows: “Let’s inquire; we’ll see what happens and then we’ll discuss the decisions to be made.” More than the *a priori* analysis, it is here the analysis I have called *in vivo* that plays a cardinal role, between *a priori* and *a posteriori* analyses.

- In the type of scenarios I am trying to describe here, another principle is essential. The process of study and research (SRP_r) through which the inquiry into Q is carried out must be subject to *regular stocktaking* indicating, at time t, the state of the Herbartian schema

$$[S(X, y, Q) \rightarrow M_t] \hookrightarrow A_t^{\heartsuit}.$$

- In particular, such stocktakings must describe, in addition to the state $A_t^{\heartsuit}$. of the answer under construction, the sequence of states M_0, M_1, M_2, …, of the didactic milieu M. This latter description relates not only to the questions generated by the inquiry and the answers to these questions arrived at by the class, but also to the *means* of producing these questions and answers, that is, *the works that are tools used for this*.

- As a test, we could thus try to describe the tools used above in the short study on floating boxes: we would find there, in particular, *algebraic calculations with parameters*, as well as elements of *hydrostatics*, especially *Archimedes' principle*.

- All the work thus done by [X, y] will be preparatory to the final work of *institutionalization* carried out in written form by the class under the direction of y. This written document will constitute a formalization of the class's knowledge on the question studied—a body of knowledge that all $x \in X$ will be required to study and which may be part of the final assessment of the students.

- Of course, the pedagogical organisation should be consistent with these working principles. In particular, it is possible to imagine the class having *seminar sessions* (where the questions to be studied, the distribution of tasks between teams, etc., are discussed, and where the stocktaking mentioned above is presented) and *workshop sessions* where principal inquiries take place.

- The same principles apply to *auxiliary inquiries*, especially when they involve a question such as “Can the work *w* be useful to the inquiry into *Q*, and, if so, how?”

Thank you for your attention!